International Journal of Theoretical Physics, Vol. 12, No. 3 (1975), pp. 177–182

Generalised Magnetic Monopole for the Yang-Mills Field

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Received: 5 *July* 1974

Abstract

By analogy to the magnetic monopole first proposed by Dirac (1931), a generalised magnetic monopole, being a source of the Yang-Mills field, is constructed. The gauge invariance and the rotational symmetry lead in a natural way to the quantisation not only of the electric charge but also of the hypercharge number Y. It is shown that the generalised magnetic charges are not arbitrary, and some restrictions on their values are deduced.

1.

The problem of the existence of the magnetic monopole, first formulated by Dirac (1931), has since been investigated by many authors. One of the main results of the monopole theory, namely the quantisation of the electric charge as a direct consequence of the existence of at least one magnetic monopole in nature, can be deduced by quite simple considerations concerning the rotational invariance of the magnetic field generated by this monopole (Schwinger, 1966). One then obtains Schwinger's condition

$$eg = 0, \pm 1, \pm 2, \ldots \tag{1.1}$$

rather than Dirac's, with

$$eg = 0, \pm \tfrac{1}{2}, \pm 1, \ldots \tag{1.1a}$$

Here e is the electric charge and g is the magnetic charge of the monopole; we make use of the natural system of units in which $\hbar = c = 1$.

Let us consider briefly the simplest method of obtaining relation (1.1), given by Peres in (1968).

Consider a homogeneous, static magnetic field B, it is obviously translation-invariant. The gauge-invariant translation generators should have the following form:

$$P'_j = p_j - eA_j, \qquad j, k, \ldots = 1, 2, 3 \tag{1.2}$$

with $p_j = -i\partial/\partial x_j$.

But these generators do not obey the canonical commutation relations; as a matter of fact, we have

$$[P'_j, P'_k] = -ie\epsilon_{jkl}B_l \tag{1.3}$$

It is easy to see that the right generators are

$$P_j = P'_j - \tfrac{1}{2}e\,\epsilon_{jkm}x_k B_m \tag{1.4}$$

These generators are manifestly gauge-invariant and verify

$$[P_j, P_k] = 0 \tag{1.5}$$

Consider now a spherically symmetric magnetic field generated by a magnetic point charge g:

$$B_k = g\frac{x_k}{r^3} \tag{1.6}$$

The classical gauge-invariant generators of the three-dimensional rotations should be

$$J'_k = \epsilon_{klm}x_l(p_m - eA_m) \tag{1.7}$$

but again, instead of the right commutation relations of the $O(3)$ group they verify

$$[J'_k, J'_l] = i\epsilon_{klm}(J'_m - egn_m) \tag{1.8}$$

where $n_m = x_m/r$.

In order to obtain the right commutation relations one should introduce the modified generators:

$$J_k = J'_k + egn_k \tag{1.9}$$

These generators are gauge-invariant and verify

$$[J_k, J_m] = i\epsilon_{kmn}J_n \tag{1.10}$$

Multiplying (1.9) by n^k we obtain

$$eg = n^k J_k \tag{1.11}$$

and the eigenvalues of this expression in a representation in which n and J are diagonal become

$$eg = 0, \pm 1, \pm 2, \ldots \tag{2.1}$$

2.

Consider now a gauge field generated by some compact, semi-simple Lie group G, dim $G = N$. The equations of this field are:

$$\partial^{\mu} F^{a}_{\mu\nu} + C^{a}_{bc} A^{b\mu} F^{c}_{\mu\nu} = 0 \tag{2.1}$$

and

$$\partial^{\mu} \check{F}^{a}_{\mu\nu} + C^{a}_{bc} A^{b\mu} \check{F}^{c}_{\mu\nu} = 0 \tag{2.1a}$$

$\mu, \nu = 0, 1, 2, 3; a, b, \ldots = 1, 2, \ldots N$. Here

$$F^{a}_{\mu\nu} = \partial_{\mu} A_{\nu}{}^{a} - \partial_{\nu} A_{\mu}{}^{a} + C^{a}_{bc} A_{\mu}{}^{b} A_{\nu}{}^{c} \tag{2.2}$$

$$\check{F}^{a}_{\mu\nu} = \tfrac{1}{2} \epsilon_{\mu\nu\kappa\lambda} F^{a}_{\kappa\lambda} \tag{2.3}$$

and C^{a}_{bc} are the structure constants of the Lie group G.

Introducing the 'electric' and 'magnetic' components of the field by

$$E_{k}{}^{a} = F^{a}_{0k}, B_{k}{}^{a} = \tfrac{1}{2} \epsilon_{klm} F^{a}_{lm} = \check{F}^{a}_{0k} \tag{2.4}$$

and choosing the gauge in which

$$A_{0}{}^{a} = 0, \qquad \partial^{i} A_{i}{}^{a} = 0 \tag{2.5}$$

we can rewrite equations (2.1) and (2.1a) in the following form:

$$\frac{\partial \vec{E}^{a}}{\partial t} = \operatorname{rot} \vec{B}^{a} + C^{a}_{bc} (\vec{A}^{b} \wedge \vec{B}^{c}) \tag{2.6}$$

$$-\frac{\partial \vec{B}^{a}}{\partial t} = \operatorname{rot} \vec{E}^{a} + C^{a}_{bc} (\vec{A}^{b} \wedge \vec{E}^{c}) \tag{2.6a}$$

We see then, that the divergences of both $\vec{E}^{a}$ and $\vec{B}^{a}$ are non-null. Being non-linear, the gauge field behaves as a source of current itself and, what is most important, it does create a current corresponding to both the 'electric' and 'magnetic' components. Both these currents are conserved, of course. Returning to the four-dimensional formalism and taking the divergence of (2.1) and (2.1a), we obtain

$$\partial^{\nu} (C^{a}_{bc} A^{b\mu} F^{c}_{\mu\nu}) = 0, \qquad \partial^{\nu} (C^{a}_{bc} A^{b\mu} \check{F}^{c}_{\mu\nu}) = 0 \tag{2.7}$$

Having this in mind, one is naturally tempted to introduce the external sources for both components, i.e.

$$\begin{aligned} \partial^{\mu} F^{a}_{\mu\nu} &= J_{\nu}{}^{a} - C^{a}_{bc} A^{b\mu} F^{c}_{\mu\nu} \\ \partial^{\mu} \check{F}^{a}_{\mu\nu} &= \check{J}_{\nu}{}^{a} - C^{a}_{bc} A^{b\mu} \check{F}^{c}_{\mu\nu} \end{aligned} \tag{2.8}$$

These external currents are not conserved by themselves, but only with the field-produced ones:

$$\partial^{\nu} (J_{\nu}{}^{a} - C^{a}_{bc} A^{b\mu} F^{c}_{\mu\nu}) = 0, \qquad \partial^{\nu} (\check{J}_{\nu}{}^{a} - C^{a}_{bc} A^{b\mu} \check{F}^{c}_{\mu\nu}) = 0 \tag{2.9}$$

Let us now generalise the considerations concerning the translational and rotational invariance mentioned in Section 1. Consider a homogeneous, translation-invariant gauge field of the type

$$\vec{E}^a = 0, \qquad \vec{B}^a = \overrightarrow{\text{Const.}} \tag{2.10}$$

The gauge-invariant generator of the translations is

$$P_k' = p_k - Q_a A_k^a \tag{2.11}$$

with Q_a now being operators obeying the Lie algebra commutation relations:

$$[Q_a, Q_b] = C_{ab}^c Q_c \tag{2.12}$$

Again, the generators P_i' do not obey the right commutation rules:

$$[P_i', P_j'] = -i\epsilon_{ijk} Q_a B_k^a \tag{2.13}$$

whereas the right expression is:

$$P_i = P_i' - \tfrac{1}{2}\epsilon_{ijk} x_j Q_a B_k^a \tag{2.14}$$

Of course, the Q_a's do not commute, but in this particular case both A_k^a and B_k^a are factorisable, i.e. of the form

$$A_k^a = f^a A_k \tag{2.15}$$

f^a being constant c-numbers, therefore vanishing in any antisymmetric combination.

The same is valid for the case of the generalised magnetic monopole, whose field is of the form

$$B_k^a = g^a \frac{x_k}{r^3} \tag{2.16}$$

There exists another exact solution of the Yang-Mills field equations, in the particular case of the $SU(2)$ gauge group, displaying the essential properties of a magnetic monopole. This solution, which is of the form

$$A_0^a = 0, \qquad A_i^a = \frac{\epsilon_{aij} x_j}{r^2}, \qquad B_i^a = -\frac{x_a x_i}{r^4} \tag{2.17}$$

has been discussed by the author (Kerner, 1970) and by Joseph (1972). This solution is not spherically symmetric and cannot justify the analogy with the electric charge quantisation described above. Its symmetry properties are discussed in Joseph (1972).

Here we will suppose the existence of the monopole described by solution (2.16), corresponding to any compact and semi-simple gauge group, of dimension higher than 3. Continuing the analogy, we have to introduce the gauge-invariant rotation generators

$$J_k' = \epsilon_{klm} x_l (p_m - Q_a A_m^a) \tag{2.18}$$

and we see again

$$[J_i', J_k'] = i\epsilon_{ikj}(J_j' - Q_a g^a n_j) \tag{2.19}$$

But if we put

$$J_k = J'_k + Q_a g^a n_k \tag{2.20}$$

we now obtain the correct commutation relations

$$[J_i, J_k] = i\epsilon_{ikl} J_l \tag{2.21}$$

Here also the result is not altered in spite of the fact that the Q_a's do not commute, because our field is factorisable.

Then, in analogy with (1.1), we have

$$Q_a g^a = n^k J_k \tag{2.22}$$

Relation (2.22) tells us that in the representation in which the operator $n^k J_k$ is diagonal, the operator $Q_a g^a$ is diagonal too. On the other hand we know that this can generally be achieved only if the coefficients g^a are chosen in such a way that the linear combination $Q_a g^a$ belongs to the Cartan subalgebra of the Lie algebra of G. The eigenvalues of the operator $Q_a g^a$ are then equal by virtue of (2.22) to $0, \pm 1, \pm 2, \ldots$.

3.

The interpretation of this result depends of course on the choice of the gauge group G, and puts restrictions on the choice of the components of the generalised magnetic monopole. Let us take for example the unitary group $SU(3)$, whose Cartan subalgebra is of dimension 2. Let us choose the basis of this subalgebra in the usual way, namely

$$Y = \sqrt{(\tfrac{1}{3})}\lambda_8, \qquad T_3 = \tfrac{1}{2}\lambda_3 \tag{3.1}$$

Y meaning the hypercharge, T_3 the third component of the isospin. The electric charge is given by the formula

$$Q_{\text{el}} = T_3 + \tfrac{1}{2} Y \tag{3.2}$$

Let us call the corresponding components of the generalised monopole g_Y and g_{T_3}. Then, because the operator T_3 takes on the values 0, 1, . . ., the following conclusions are obvious concerning the possible values of g_Y and g_{T_3}:

	Could exist	Cannot exist
1.	$g_Y = 1, g_{T_3} = 0$	–
2.	–	$g_Y = 0, g_{T_3} = 1$
3.	$g_Y = \frac{1}{2}, g_{T_3} = 1$	–
4.	–	$g_Y = 1, g_{T_3} = 1$

The first generalised monopole is therefore responsible for the quantisation of the hypercharge, the third one yields the quantisation of the electric charge;

a monopole producing the effect of the quantisation of both these numbers at the same time does not exist.

To conclude, let us remark that it is easy to see that were there generalised dyons in nature (i.e. particles being at the same time 'electric' and 'magnetic' monopoles, in the sense of (2.4)), the above restrictions are valid for the magnetic monopole coefficients g_a, whereas the 'electric'-component coefficients q_a can be arbitrary, not necessarily in the Cartan subalgebra.

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